SOLUTION OF AN INVERSE PROBLEM OF CAVITATIONAL FLOW AROUND A CURVILINEAR ARC

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ABSTRACT: The problem of the structure of the cavitational flow around a curvilinear arc in accordance with the Ryabushinskii scheme with a given velocity distribution is considered. The inverse problem was formulated and solved for the first time for the case of separated flow in accordance with the Kirchhoff scheme in an unbounded stream by G. G. Tumashev [1], and by G. N. Pykhteev [2] for an arc in a channel. It was also the latter who solved the inverse problem of flow around an arc in accordance with the scheme of Gil'berg and Efros [3].

1. Let us consider cavitational flow of a plane stream of an ideal incompressible fluid around a symmetric curvilinear arc L in accordance with the Ryabushinskii scheme with a mirror, as shown in Fig. 1.

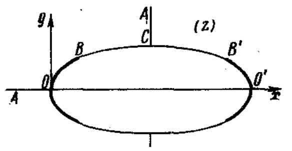

Fig. 1

We shall use the letters V, V_0, $2S_0$, and φ_0 to denote the velocity of the undisturbed flow, the free-jet velocity, the length of the arc L, and the value of the velocity potential at the separation points, respectively.

Let the distribution of the modulus of the velocity be given on the arc in the form of a function of the arc abscissa,

$$V = V_0 f(s) \qquad (s = S / S_0, \qquad 0 \leqslant s \leqslant 1),$$

The function $f(s)$ is assumed to be single-valued, positive, and satisfies a Hölder condition and the conditions $f(0) = 0$, $f(1) = 1$.

It is required to construct the form of the contour L and the form of the free jets, also to find the resistance of the arc. On the strength of symmetry, we shall consider only the flow in the second quadrant of the physical z-plane.

2. We shall consider the complex velocity potential $W = \varphi + i\psi$ at point C, and also the stream function ψ on the streamline AOBCB'O'A, to be equal to zero. Then the region of variation W will be the second quadrant of the plane. The function $W(\zeta)$ mapping the region of variation W onto the first quadrant of the auxiliary variable $\zeta = \xi + i\eta$ with correspondence of the points shown in Figs. 1, 2, and 3 is easily found,

$$W(\zeta) = -\frac{\varphi_0 \sqrt{1+a^2}}{\sqrt{\zeta^2+a^2}}. \tag{2.1}$$

On the arc L

$$\varphi = -\varphi_1 + V_0 S_0 \int_0^s f(s)\,ds, \qquad \varphi_1 = \frac{\varphi_0 \sqrt{1+a^2}}{a}. \tag{2.2}$$

From (2.1) and (2.2), we find the relationship $S(\xi)$ establishing the correspondence between points of the arc L and the points of the segment $[-1, 1]$ of the ξ-axis of the ζ-plane:

$$-\frac{\varphi_0 \sqrt{1+a^2}}{\sqrt{a^2+\xi^2}} = -\varphi_1 + V_0 S_0 \int_0^s f(s)\,ds\,. \tag{2.3}$$

We introduce the Zhukovskii function

$$F(\zeta) = \ln\left(\frac{1}{V_0}\frac{dW}{dz}\right) = \ln\frac{V}{V_0} - i\vartheta\,. \tag{2.4}$$

The function $F(\zeta)$ is real and continuous on the imaginary η-axis of the ζ-plane. We shall extend it to the entire upper half-plane. Now, $F(\zeta)$ is defined and analytic over the entire upper half-plane and satisfies the following boundary conditions:

$$\operatorname{Re} F(\xi) = \begin{cases} 0 & \text{for} \quad |\xi| > 1\,, \\ \ln f & \text{for} \quad |\xi| \leqslant 1\,. \end{cases}$$

As is well known, the problem of reconstructing the real part of a function which is analytical in the upper half-plane and given on the real axis is solved by a Cauchy type integral (refer, for example, [4])

$$F(\zeta) = \frac{1}{\pi i}\int_{-1}^{1} \frac{\ln f(t)}{t-\zeta}\,dt. \tag{2.5}$$

The function $z(\zeta)$ is now determined from the relationships (2.1), (2.4), and (2.5):

$$z(\zeta) = \sqrt{1+a^2}\,\frac{\varphi_0}{V_0}\int \frac{\zeta}{(\zeta^2+a^2)^{3/2}} \exp\left[-F(\zeta)\right] d\zeta + \text{const}. \tag{2.6}$$

Separating the real and imaginary parts of (2.6), and passing to the limit with $\zeta \to \xi$, we obtain:

the equation for the contour,

$$x = \frac{\varphi_0 \sqrt{1+a^2}}{V_0}\int_0^\xi \frac{\xi}{f[s(\xi)]\,(\xi^2+a^2)^{3/2}} \cos I(\xi)\,d\xi\,,$$

$$y = \frac{\varphi_0 \sqrt{1+a^2}}{V_0}\int_0^\xi \frac{\xi}{f[s(\xi)]\,(\xi^2+a^2)^{3/2}} \sin I(\xi)\,d\xi$$

$$\left(I(\xi) = \frac{1}{\pi}\int_{-1}^{1} \frac{\ln f}{t-\xi}\,dt, \qquad |\xi| \leqslant 1\right); \tag{2.7}$$

the equation for the jets,

$$x = x_0 + \frac{\varphi_0 \sqrt{1+a^2}}{V_0}\int_1^\xi \frac{\xi}{(\xi^2+a^2)^{3/2}} \cos\Phi(\xi)\,d\xi,$$

$$y = y_0 + \frac{\varphi_0 \sqrt{1+a^2}}{V_0}\int_1^\xi \frac{\xi}{(\xi^2+a^2)^{3/2}} \sin\Phi(\xi)\,d\xi$$

$$\left(\Phi(\xi) = \frac{1}{\pi}\int_{-1}^{1} \frac{\ln f}{t-\xi}\,dt, \qquad |\xi| \geqslant 1\right). \tag{2.8}$$

Here x_0, y_0 are the coordinates of the separation point of a jet.

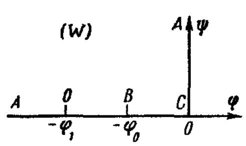

Fig. 2

3. Now, we shall define the parameters included in the solution. It is necessary that the following conditions be satisfied in the solution of the problem:

$$\left(\frac{dW}{dz}\right)_{\zeta=ai} = V_\infty, \qquad [\varphi(s)]_{s=1} = -\varphi_0.$$

It can be shown that they are transformed to the form

$$\ln \frac{V_\infty}{V_0} = G(a). \qquad (3.1)$$

Here

$$G(a) = \frac{a}{\pi} \int_{-1}^{1} \frac{\ln f}{t^2 + a^2} dt, \qquad -\varphi_0 = -\varphi_1 + V_0 S_0 \int_0^1 f(s)\, ds. \quad (3.2)$$

These equalities are the equations for determining the parameters a and φ_0 for the given cavitation number $Q = V_0^2/V_\infty^2 - 1$ and the given length of the arc S_0.

4. Making use of the Bernoulli integral, the expression for the resistance can be written in the form of

$$R = \rho V_0^2 S_0 \int_0^1 [1 - f^2(s)] \sin\theta(s)\, ds$$

or, going from the variable s to the variable ξ,

$$R = \rho V_0^2 S_0 \int_0^1 \frac{1 - f^2[S(\xi)]}{f[S(\xi)]} \frac{\xi}{(\xi^2 + a^2)^{3/2}} \sin I(\xi)\, d\xi. \quad (4.1)$$

5. Let us consider a special case. Let the function $f(s)$ be of the form

$$f(s) = \frac{As}{\sqrt{B^2 - s^2}} \qquad (A,\ B = \text{const}).$$

On substituting this expression in (2.3) and (3.2), we obtain

$$-\frac{\varphi_0 \sqrt{1 + a^2}}{\sqrt{\xi^2 + a^2}} = -\frac{\varphi_0 \sqrt{1 + a^2}}{a} + V_0 S_0 \cdot A\,(B - \sqrt{B^2 - s^2}),$$

$$-\varphi_0 = \frac{a V_0 S_0}{\sqrt{1 + a^2} - a} A\,(B - \sqrt{B^2 - 1}).$$

Let $A = a$, $B = (1 + d^2)^{1/2}$. Then

$$\varphi_0 = a^2 V_0 S_0, \quad s(\xi) = \frac{\xi \sqrt{1 + a^2}}{\sqrt{a^2 + \xi^2}}, \qquad f[s(\xi)] = \xi. \quad (5.1)$$

The values of the integrals $I(\xi)$, $\Phi(\xi)$, and $G(a)$ are found in [3],

$$I(\xi) = \frac{\pi}{2} - \frac{\pi}{4} N(\xi), \quad \Phi(\xi) = \frac{1}{2} \arcsin \frac{1}{\xi} + \frac{\pi}{4} N\left(\frac{1}{\xi}\right), \quad (5.2)$$

$$G(a) = \frac{1}{2} \ln \frac{\sqrt{1 + a^2} - 1}{2a} + \frac{1}{\pi}\left(\operatorname{arc\,tg} \frac{1}{a}\right) \ln a +$$
$$+ \frac{1}{\pi} L\left(\operatorname{arc\,tg} \frac{1}{a}\right) + \frac{1}{\pi} L(\operatorname{arc\,tg} a), \qquad (5.3)$$

$$N(\tau) = \frac{2}{\pi^2} \int_0^\tau \frac{1}{\tau} \ln \frac{1 + \tau}{1 - \tau} d\tau, \qquad L(\tau) = -\int_0^\tau \ln\cos\tau\, d\tau. \quad (5.4)$$

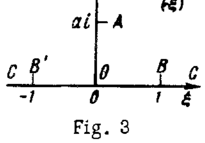

Fig. 3

Substituting (5.1) in (2.7) and (4.1) and (5.2) in (2.8), we obtain:
the equation of the contour

$$x = a^2 \sqrt{1 + a^2} S_0 \int_0^\xi \frac{1}{(a^2 + \xi^2)^{3/2}} \sin\left[\frac{\pi}{4} N(\xi)\right] d\xi,$$

$$y = a^2 \sqrt{1 + a^2} S_0 \int_0^\xi \frac{1}{(a^2 + \xi^2)^{3/2}} \cos\left[\frac{\pi}{4} N(\xi)\right] d\xi, \qquad |\xi| \leqslant 1;$$

the equation of the jets

$$x = x_0 +$$
$$+ a^2 \sqrt{1 + a^2} S_0 \int_1^\xi \frac{\xi}{(a^2 + \xi^2)^{3/2}} \cos\left[\frac{1}{2} \arcsin \frac{1}{\xi} + \frac{\pi}{4} N\left(\frac{1}{\xi}\right)\right] d\xi,$$

$$y = y_0 +$$
$$+ a^2 \sqrt{1 + a^2} S_0 \int_1^\xi \frac{\xi}{(a^2 + \xi^2)^{3/2}} \sin\left[\frac{1}{2} \arcsin \frac{1}{\xi} +\right.$$
$$\left. + \frac{\pi}{4} N\left(\frac{1}{\xi}\right)\right] d\xi \quad |\xi| \geqslant 1;$$

the resistance of the arc

$$R = \rho V_0^2 S_0 a^2 \sqrt{1 + a^2} \int_0^1 \frac{1 - \xi^2}{(a^2 + \xi^2)^{3/2}} \cos\left[\frac{\pi}{4} N(\xi)\right] d\xi.$$

Bearing (5.4) in mind, we rewrite Eq. (3.1),

$$\ln \frac{V_\infty}{V_0} = \frac{1}{2} \ln \frac{\sqrt{1 + a^2} - 1}{2a} + \frac{1}{\pi}\left(\operatorname{arc\,tg} \frac{1}{a}\right) \ln a +$$
$$+ \frac{1}{\pi} L\left(\operatorname{arc\,tg} \frac{1}{a}\right) + \frac{1}{\pi} L(\operatorname{arctg} a).$$

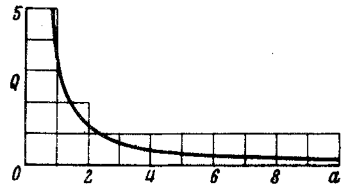

Fig. 4

The length and width of a cavity are found by the following formulas:

$$l = 2\left\{x_0 + a^2 \sqrt{1 + a^2} S_0 \int_1^\infty \frac{\xi}{(a^2 + \xi^2)^{3/2}} \times\right.$$
$$\left. \times \cos\left[\frac{1}{2} \arcsin \frac{1}{\xi} + \frac{\pi}{4} N\left(\frac{1}{\xi}\right)\right] d\xi\right\},$$

$$h = 2\left\{y_0 + a^2 \sqrt{1 + a^2} S_0 \int_1^\infty \frac{\xi}{(a^2 + \xi^2)^{3/2}} \times\right.$$
$$\left. \times \sin\left[\frac{1}{2} \arcsin \frac{1}{\xi} + \frac{\pi}{4} N\left(\frac{1}{\xi}\right)\right] d\xi\right\}.$$

Figure 4 shows the curve of the cavitation number Q versus the parameter a.

6. We shall now give the dependence of the velocity on the arc in parametric form,

$$V = V_0 F_1(u), \qquad S = S_0 F_2(u),$$

where $F_1(u)$ and $F_2(u)$ are single-valued positive functions $u \in [4_1, u_2]$ satisfying a Hölder condition and the conditions

$$F_1(u_1) = F_2(u_1) = 0, \qquad F_1(u_2) = F_2(u_2) = 1.$$

The given problem can be reduced to the preceding one in the following manner:

$$f(s) = f\ [s(\xi)] = F_1[u(\xi)].$$

The relationship $u(\xi)$ is found from the equation

$$-\frac{\varphi_0 \sqrt{1 + a^2}}{\sqrt{\xi^2 + a^2}} = -\varphi_1 + V_0 S_0 \int_{u_1}^{u} F_1(u) F_2'(u)\, du. \qquad (6.1)$$

The equation for finding φ_0 is rewritten as

$$\varphi_0 = -\varphi_1 + V_0 S_0 \int_{u_1}^{u_2} F_1(u) F_2'(u)\, du. \qquad (6.2)$$

For example, let

$$F_1(u) = \left(\frac{1-u}{1+u}\right)^{1/2}, \qquad F_2(u) = -\frac{1}{A}\int_1^u \frac{(u+1)\, u du}{(u^2+\beta^2)^{3/2}\sqrt{1-u^2}}$$

$$\left(A = \int_0^1 \frac{(u+1)\, u du}{(u^2+\beta^2)^{3/2}\sqrt{1-u^2}}, \ \alpha, \ m, \ \beta - \text{const}\right).$$

Substituting the expression for $F_1(u)$ and $F_2(u)$ in formulas (6.1) and (6.2), we obtain the equations for finding the relationships $u(\xi)$ and φ_0, respectively:

$$\frac{\varphi_0\sqrt{1+a^2}}{\sqrt{\xi^2+a^2}} = \frac{\varphi_0\sqrt{1+a^2}}{a} + \frac{2V_0S_0}{A}\left(\frac{1}{\sqrt{\beta^2-u^2}} - \frac{1}{\sqrt{\beta^2-1}}\right),$$

$$\varphi_0\left(1 - \frac{\sqrt{1+a^2}}{a}\right) = \frac{2V_0S_0}{A}\left(\frac{1}{\beta} - \frac{1}{\sqrt{\beta^2-1}}\right).$$

We set $\beta = (1+a^2)^{1/2}$. Then

$$\varphi_0 = \frac{2V_0S_0}{A\sqrt{1+a^2}}, \quad u(\xi) = \sqrt{1-\xi^2}, \quad f[s(\xi)] = \left(\frac{1-\sqrt{1-\xi^2}}{1+\sqrt{1-\xi^2}}\right)^{1/2}.$$

We shall write the values of the integrals [3] $I(\xi)$ and $\Phi(\xi)$ in this case,

$$I(\xi) = {}^1/_2\pi\,(\operatorname{sign}\xi), \qquad \Phi(\xi) = 2\operatorname{arc\,tg}(\xi - \sqrt{\xi^2-1}). \quad (6.3)$$

Substituting (6.3) in (2.7), we obtain the equation for the contour:

$$x = 0, \qquad y = \frac{\varphi_0\sqrt{1+a^2}}{V_0}\int_0^\xi \frac{1-\sqrt{1-\xi^2}}{(\xi^2+a^2)^{3/2}}\, d\xi.$$

Thus, we have obtained the flow around a plate arranged normal to the flow.

The equation for the jet and the resistance of the contour are found from formulas (2.8) and (4.1), respectively. On computing the value of $G(a)$ for the given case and substituting it in formula (3.1), we obtain an equation for determining the parameter a.

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